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Vector Measures on the Logic of J-projections of a Krein Space

Marjan Matvejchuk[1,3] and Anna Ionova[2]

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Let $\mathcal{H}$ be a Hilbert space with an inner product $(.,.)_{\mathcal{H}}$. In Jajte, R., and Paszkiewicz, A. (1978, Vector measure on the closed subspaces of a Hilbert space, *Studia Mathematica* **63**, 229–251), the $\mathcal{H}$-measure on the logic of all orthogonal projections on $\mathcal{H}$ was studied. We examine the $\mathcal{H}$-measure on the hyperbolic logic of all J-projections on a Krein space.

KEY WORDS: quantum logic; Hilbert space; indefinite metric space; projection; measure.

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1. INTRODUCTION

One of the basic problems related to the propositional calculus approach to the foundations of quantum mechanics is the description of measures on the set of experimentally verifiable propositions regarding a physical system. The set of propositions form an orthomodular partially ordered set, where the order is induced by a relation of implication, and is called a *quantum logic*.

An important interpretation of a quantum logic is the set $B(H)^{\text{pr}}$ of all orthogonal projections on a Hilbert space H. The problem of the construction of a quantum field theory sometimes leads to an indefinite metric space (Dadashyan and Horujy, 1983). In this case, the set $\mathcal{P}$ of all J-orthogonal projections serves to be an analog to the logics $B(H)^{\text{pr}}$. There is an indefinite analog (Matvejchuk, 1997) to the remarkable Gleason's theorem. In this paper a vector measure on the quantum logic $\mathcal{P}$ is studied for the first time.

Let H be a (complex) Hilbert space with an inner product $(.,.)_H$. Fix a self-adjoint symmetry operator J ($J = J^* = J^{-1}$, $J \neq \pm I$). The form

[1] Kazan State University, 420008, Kazan, Kremlevskaja str., Russia.
[2] Ulyanovsk State Pedagogik University, 423700 Ulyanovsk, Tolstogo str. 87, Apartm 33, Russia.
[3] To whom correspondence should be addressed at 420043, Kazan, Czechov str. 31, Apartm 109, Russia; e-mail: marjan.matvejchuk@ksu.ru.

$[x, y] := [x, y]_H := (Jx, y)_H$ is said to be an *indefinite metric*, and H is said to be the *Krein space* (= J-*space*) (see Azizov and Iokhvidov, 1986). Put $Q^+ := \frac{1}{2}(I + J)$ and $Q^- := I - Q^+$, $H^+ := Q^+H$, $H^- := Q^-H$. The representation $H = H^+[\dot{+}]H^-$ is said to be a *canonical decomposition* of H. Without loss of generality we can assume $\dim H^+ \leq \dim H^-$.

A vector $x \in H$ is *neutral* (*positive, nonnegative, negative, nonpositive*), if $[x, x] = 0$ ($[x, x] > 0$, $[x, x] \geq 0$, $[x, x] < 0$, $[x, x] \leq 0$, respectively). Let us denote by β^0 (β^{++}, β^+, β^{--}, β^-) the set of all neutral (positive, nonnegative, negative, nonpositive, respectively) vectors. Put $\Gamma^+ \equiv \{f \in H : [f, f] = 1\}$ and $\Gamma^- \equiv \{f \in H : [f, f] = -1\}$. It is clear that $J\Gamma^\pm = \Gamma^\pm$. The set $\Gamma := \Gamma^+ \cup \Gamma^-$ is an indefinite analogy of the unit sphere S of H.

An operator $A \in B(H)$ is J-*positive* (J-*negative*) if $[Ax, x] \geq 0$ ($[Ax, x] \leq 0$), for all $x \in H$. Note that $A \in B(H)$ is J-positive (negative) if and only if $JA \geq 0$ ($JA \leq 0$, respectively). Let $A \in B(H)$. The operator $A^\# := JA^*J$ ($\equiv [Ax, y] = [x, A^\# y]$, for all x, $y \in H$) is said to be J-*adjoint* to A. Write $\Re A := \frac{1}{2}(A + A^*)$, $\Im A := \frac{1}{2i}(A - A^*)$, for all $A \in B(H)$. Set $L_1 := \{A \in B(H) : \mathrm{tr}(|\Re A| + |\Im A|) < +\infty\}$.

Let $\mathcal{P} := \{p \in B(H) :\ p^2 = p,\ [px, y] = [x, py],\ \forall x, y \in H\}$ and let $\mathcal{P}^+$ ($\mathcal{P}^-$) be the set of all J-positive (J-negative, respectively) projections from $\mathcal{P}$. For any $p \in \mathcal{P}$, there exists a (nonunique!) representation $p = p_+ + p_-$, where $p_+ \in \mathcal{P}^+$ and $p_- \in \mathcal{P}^-$. Any one-dimensional J-projection has the form $p_f := [f, f][., f]f$, $f \in \Gamma$ and $(p_f)^* = p_{Jf}$. If $f \in \Gamma^+$ ($f \in \Gamma^-$) then $p_f, p_{Jf} \in \mathcal{P}^+$ ($\in \mathcal{P}^-$, respectively). If $e_\pm \in H^\pm \cap S$ and if complex numbers α, β are such that $|\alpha|^2 - |\beta|^2 = 1$ then $(., \alpha e_+ - \beta e_-)(\alpha e_+ + \beta e_-) \in \mathcal{P}^+$. (If $|\alpha|^2 - |\beta|^2 = -1$ then $-(., \alpha e_+ - \beta e_-)(\alpha e_+ + \beta e_-) \in \mathcal{P}^-$.) One can prove that: $\dim((p_f)_+ H) = \frac{1}{2}([f, f] + 1)$, for all $f \in \Gamma$; $p_f \in B(H)^{\mathrm{pr}} \Leftrightarrow f \in S \cap \Gamma$ ($= S \cap (H^+ \cup H^-)$). Let $q_f := \|f\|^{-2}(., f)_H f$, for all $f \neq 0$. Note $q_f \in B(H)^{\mathrm{pr}}$.

Any sum $p = \Sigma p_\iota$ where $p, p_\iota \in \mathcal{P}$ for all i, and $p_\iota p_j = 0$ if $i \neq j$ is said to be a *decomposition* of p (the sum understood in the weak sense).

Let $\mathcal{H}$ be a complex Hilbert space. A function $\xi : \mathcal{P} \to \mathcal{H}$ is called an $\mathcal{H}$-*measure* if $\xi(p) = \Sigma \xi(p_i)$ for any decomposition $p = \Sigma p_i$ (the series being weakly convergent).

An $\mathcal{H}$-measure ξ is said to be: *semiconstant* if $\dim H^+ < +\infty$ and there is a vector $h_0 \in \mathcal{H}$, $h_0 \neq 0$ such that $\xi(p) = \dim(p_+)h_0$, for all $p \in \mathcal{P}$; *linear* if there exists a linear operator $\hat{\xi} : B(H) \to \mathcal{H}$ such that $\hat{\xi}(p) = \xi(p)$, for all $p \in \mathcal{P}$; *w-linear* if for any $h \in \mathcal{H}$ there is an operator $M_h \in L_1$ such that $(\xi(q), h)_{\mathcal{H}} = \mathrm{tr}(qM_h)$, for all $q \in \mathcal{P}$; *bounded* if $K \equiv \sup_{q \in \mathcal{P}} \left\{ \frac{\|\xi(q)\|}{\|q\|} \right\} < +\infty$; *w-bounded* if $K_h \equiv \sup_{q \in \mathcal{P}} \left\{ \frac{|(\xi(q), h)_{\mathcal{H}}|}{\|q\|} \right\} < +\infty$, for all $h \in \mathcal{H}$; if $\mathcal{H} = C$ ($= R$) the C- (R)-measure is said to be a *complex* (*real*) measure. Here C (R) is the set of all complex (real, respectively) numbers.

Remark 1.
Any semiconstant measure is bounded but is not a linear measure; any complex measure $\xi(\cdot)$ is the sum of real measures $\Re\xi(\cdot)$ and $\Im\xi(\cdot)$.

The main results for the real measure are the following:

Theorem 2.3 of Matvejchuk (1997). *Let H be a Krein space,* $\dim H \geq 3$, $\dim H^+ \leq \dim H^-$ *and let $\mu : \mathcal{P} \to R$ be a real measure. The measure μ is bounded if and only if there exist a unique J-self-adjoint trace-class operator M and unique number $c \in R$ such that* $\mu(q) = \mathrm{tr}(qM) + c\dim(q_+H)$, for all $q \in \mathcal{P}$. If $\dim H^+ = +\infty$, then $c = 0$ $(0.\infty \equiv 0)$.

Theorem 2.1 of Matvejchuk (1997). *Let H be a Krein space,* $\dim H = +\infty$. *Then any real measure $\mu : \mathcal{P} \to R$ is bounded.*

Let ξ be a bounded $\mathcal{H}$-measure. By Theorem 2.3 of Matvejchuk (1997), for all $h \in \mathcal{H}$ there exist a unique $M_h \in L_1$ and a unique number c_h, such that

$$\xi_h(q) := (\xi(q), h)_{\mathcal{H}} = \mathrm{tr}(qM_h) + c_h \dim(q_+H), \forall q \in \mathcal{P}. \quad (1)$$

Let $\mathcal{Z} \subset H$. It is well known that the inequality $\sup_{z\in\mathcal{Z}}\{||z||\} < +\infty$ is equivalent to $\sup_{z\in\mathcal{Z}}\{|(z,h)_H|\} < +\infty$, for all $h \in H$. Hence we have

Proposition 1. *The $\mathcal{H}$-measure ξ is bounded if and only if ξ is w-bounded. If* $\dim H = +\infty$ *then every $\mathcal{H}$-measure is bounded.*

Lemma 1. *For any $M \in B(H)$,*

$$\|M\| \leq 8 \sup_{f\in S\cap\beta^{--}} \{|(Mf, f)|\}.$$

Proof: Let $\phi \in S$. It is clear that $\sup_{f\in S\cap\beta^{--}}(q_\phi f, f) = \sup_{f\in S\cap\beta^{-}}(q_\phi f, f)$. Let $e^+ \in H^+ \cap S$, $e^- \in H^- \cap S$ be such that $\phi = \alpha e^+ + \beta e^-$. Here $\alpha \geq 0$, $\beta \geq 0$. Let f denote the vector $1/\sqrt{2}(e^+ + e^-)$. Then $f \in \beta^0$ $(\subset \beta^-)$ and

$$2(q_\phi f, f)_H \geq (\alpha^2 + \beta^2) = 1. \quad (2)$$

1) Let first $M = M^* \in B(H)$ and let
   a) $\dim H^+ = \dim H^- = 1$. Without loss of generality we can consider the two cases:
      i) $M = (b+a)q_\phi + aq_\phi^\perp$, where $\phi, \phi^\perp \in S, (\phi, \phi^\perp)_H = 0$ and $b \geq 0$, $a \geq 0$. In this case, by (2), we have

$$\sup_{f\in S\cap\beta^{-}} \{|(Mf, f)|\} = \sup_{f\in S\cap\beta^{-}} \{b(q_\phi f, f)\} + a \geq \frac{b}{2}$$

$$+a \geq \frac{(b+a)}{2} = \frac{\|M\|}{2}. \tag{3}$$

ii) $M = (b+a)q_\phi - aq_\phi^\perp$. In this case we have

$$\sup_{f\in S\cap\beta^{--}} |(Mf, f)| = \sup_{f\in S\cap\beta^{--}} |(b+2a)(p_\phi f, f) - a|$$

$$\geq (b+2a) \sup_{f\in S\cap\beta^{--}} (q_\phi f, f) - a \geq \frac{(b+2a)}{2} - a = \frac{b}{2}.$$

If $\phi \in \beta^- \cap S$ then

$$\|M\| = (b+a) = (M\phi, \phi) = \sup_{f\in S\cap\beta^{--}} |(Mf, f)|.$$

If $\phi \in \beta^{++} \cap S$ then $\phi^\perp \in \beta^{--} \cap S$ and $|(M\phi^\perp, \phi^\perp)| = a$. Hence

$$\sup_{f\in S\cap\beta^{--}} |(Mf, f)| \geq \max\left\{\frac{b}{2}, a\right\} \geq \frac{(b+a)}{4} = \frac{1}{4}\|M\|.$$

b) Let us consider the case of a general H. It is well known that $\|M\| = \sup_{f\in S}\{|(Mf, f)|\}$. Fix $\epsilon > 0$. Let us choose $f_0 \in S$, such that $|(Mf_0, f_0)| > \|M\| - \epsilon$ and $e^+ \in S \cap H^+$, $e^- \in S \cap H^-$ such that $f_0 = \alpha e^+ + \beta e^-$. We will denote by P the orthoprojection $(., e^+)e^+ + (., e^-)e^-$. It is clear that $\|M\| \geq \|PMP\| \geq \|M\| - \epsilon$. By 1i), 1ii)

$$\sup_{f\in S\cap\beta^{--}} \{|(Mf, f)|\} \geq \sup_{f\in S\cap\beta^{--}, f\in PH} \{|(PMPf, f)|\}$$

$$\geq \frac{1}{4}\|PMP\| \geq \frac{1}{4}(\|M\| - \epsilon).$$

Hence $\sup_{f\in S\cap\beta^{--}}\{|(Mf, f)|\} \geq \frac{1}{4}\|M\|$.

2) Let now $M \in B(H)$. By (1b),

$$\sup_{f\in S\cap\beta^{--}} \{|(Mf, f)|\} \geq \max\{\sup_{f\in S\cap\beta^{--}} \{|(\Re Mf, f)|\}, \sup_{f\in S\cap\beta^{--}} \{|(\Im Mf, f)|\}\}$$

$$\geq \frac{1}{4}\max\{\|\Re M\|, \|\Im M\|\} \geq \frac{1}{8}(\|\Re M\| + \|\Im M\|)$$

$$\geq \frac{1}{8}\|M\|.$$

□

Proposition 2. *Let* $\dim H \geq 3$, $\xi : \mathcal{P} \to \mathcal{H}$ *be a bounded* $\mathcal{H}$*-measure, and let the operator* M_h *be as in (1). The function* $h \to M_h$ *is antilinear and continuous in the norm topology on* H *and in the norm operator topology.*

Proof: By definition (1), the function $h \to M_h$ is antilinear. Let $K := \sup\{\frac{\|\xi(q)\|}{\|q\|} : q \in \mathcal{P}\}$. We have

$$K\|h\| \geq \sup_{q\in\mathcal{P}} \left| \left(\frac{\xi(q)}{\|q\|}, h \right)_{\mathcal{H}} \right| \geq \sup_{f\in\Gamma} \frac{1}{\|f\|^2} |\mathrm{tr}(p_f M_h) + \frac{c_h}{2}([f, f] + 1)|$$

$$\geq \sup_{f\in\Gamma^-} \left| \mathrm{tr}\left(\frac{p_f}{\|f\|^2} M_h \right) \right| = \sup_{f\in\Gamma^-} \left| \left(J M_h \frac{f}{\|f\|}, \frac{f}{\|f\|} \right)_H \right|$$

$$= \sup_{f\in S\cap\beta^{--}} |(J M_h f, f)_H| \geq \quad \text{by Lemma 1} \quad \geq \frac{1}{8}\|J M_h\| = \frac{1}{8}\|M_h\|.$$

□

Proposition 3. *Let H be a complex Krein space. The $\mathcal{H}$-measure ξ is w-linear if and only if ξ is a bounded linear $\mathcal{H}$-measure. If $\mathcal{H}$-measure ξ is w-linear then there is a bounded linear operator $\hat{\xi} : B(H) \to \mathcal{H}$ such that $\hat{\xi}(q) = \xi(q)$ for all $q \in \mathcal{P}$.*

Proof: Let ξ be a w-linear $\mathcal{H}$-measure and let $\xi_h(q) := (\xi(q), h)_{\mathcal{H}} = \mathrm{tr}(q M_h)$, for all $q \in \mathcal{P}, h \in \mathcal{H}$. Here $M_h \in L_1$. It is clear that ξ is bounded. We will define an $\mathcal{H}$-measure on the set $B(H)^{\mathrm{pr}}$.

1) Let us define the function $\phi(q_f) := [f, f]\|f\|^{-2}\xi(p_f)$, $f \in \Gamma$. We have

$$(\phi(q_f), h)_{\mathcal{H}} = \frac{[f, f]}{\|f\|^{-2}}(\xi(q), h)_{\mathcal{H}} = \frac{[f, f]}{\|f\|^{-2}}\mathrm{tr}(p_f M_h)$$

$$= \frac{1}{\|f\|^{-2}}\mathrm{tr}([., f] f M_h) = \frac{1}{\|f\|^{-2}}\mathrm{tr}([., f] M_h f)$$

$$= \frac{1}{\|f\|^{-2}}(J M_h f, f)_H = \mathrm{tr}(q_f J M_h). \tag{4}$$

Hence for any $f, g \in \Gamma$

$$|((\phi(q_f) - \phi(q_g)), h)_{\mathcal{H}}| = \left| \frac{1}{\|f\|^{-2}}(J M_h f, f)_H - \frac{1}{\|g\|^{-2}}(J M_h g, g)_H \right|. \tag{5}$$

Let $\varphi \in S \cap \beta^0$ and let a sequence $\{f_n\} \subset \Gamma$ be such that $\|\varphi - \frac{f_n}{\|f_n\|}\| \to 0$. By (5), $\{\phi(q_{f_n})\}_1^\infty$ is a weakly fundamental sequence. There exists the weak limit $w - \lim_n \phi(q_{f_n})$. Let $\phi(q_\varphi) := w - \lim_n \phi(q_{f_n})$. By (4),

$$(\phi(q_\varphi), h)_{\mathcal{H}} = \lim_n \left(J M_h \frac{f_n}{\|f_n\|}, \frac{f_n}{\|f_n\|} \right)_H = (J M_h \varphi, \varphi)_H = \mathrm{tr}(q_\varphi J M_h).$$

2) Let now $q_1, q_2, \ldots, q_m$ be a set of mutually orthogonal one-dimensional projections from $B(H)^{\text{pr}}$. Set $\phi(\Sigma_{i=1}^m q_i) := \Sigma_{i=1}^m \phi(q_i)$. One has

$$\left(\phi\left(\sum_{i=1}^m q_i\right), h\right)_{\mathcal{H}} = \sum_{i=1}^m (\phi(q_i), h)_{\mathcal{H}} = \sum \operatorname{tr}(q_i J M_h)$$

$$= \operatorname{tr}\left(\left(\sum_{i=1}^m q_i\right) J M_h\right).$$

Hence the vector $\phi(\Sigma_{i=1}^m q_i)$ is well defined.

3) Let us consider the case $\dim H = +\infty$. Let $q \in B(H)^{\text{pr}}$, $\dim qH = +\infty$ and let p_α be a net of finite-dimensional orthogonal projections, $q = w - \lim_\alpha p_\alpha$. Write $\phi(q) := w - \lim_\alpha \phi(p_\alpha)$. For any $p \in B(H)^{\text{pr}}$ we have $|(\phi(p), h)_{\mathcal{H}}| = |\operatorname{tr}(pJM_h)| \leq \operatorname{tr}(|JM_h|)$. Hence the function $\phi : B(H)^{\text{pr}} \to \mathcal{H}$ is a bounded (i.e., $\sup_{p \in B(H)^{\text{pr}}} \|\phi(p)\| < +\infty$) $\mathcal{H}$-measure on $B(H)^{\text{pr}}$.

The operator $A := \Sigma_{i=1}^n a_i p_i$, where $p_i \in B(H)^{\text{pr}}$, and $a_i \in R$ for all i is said to be *simple*. Let $\phi(A) := \Sigma_{i=1}^n a_i \phi(p_i)$. By the equality $(\phi(A), h)_{\mathcal{H}} = \operatorname{tr}(AJM_h)$, for all $h \in \mathcal{H}$, the vector $\phi(A)$ is well defined.

Let now $A \in B(H)$ be a self-adjoint operator and let $\{A_m\}$ be a sequence of simple operators such that $\|A - A_m\| \to 0$. Let $\phi(A) := w - \lim_{n\to\infty} \phi(A_n)$. For the general case of $A \in B(H)$ we set $\phi(A) := \phi(\Re A) + i\phi(\Im A)$. By construction, $(\phi(A), h)_{\mathcal{H}} = \operatorname{tr}(AJM_h)$. Hence $(\phi(A + B), h)_{\mathcal{H}} = \operatorname{tr}((A + B)JM_h) = (\phi(A) + \phi(B), h)_{\mathcal{H}}$, for all $A, B \in B(H)$. Hence the operator $\phi : B(H) \to \mathcal{H}$ is linear. By Corollary 1 of Jajte and Paszkiewicz (1978), the linear operator ϕ is bounded. For any $q \in \mathcal{P}$ we have

$$(\xi(q), h)_{\mathcal{H}} = \operatorname{tr}(qM_h) = \operatorname{tr}(qJJM_h) = (\phi(qJ), h)_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

Set $\hat{\xi}(\cdot) := \phi(\cdot J)$. Then $\hat{\xi}(q) = \phi(qJ) = \xi(q)$, for all $q \in \mathcal{P}$.

Conversely, let ξ be a bounded linear $\mathcal{H}$-measure and let a linear operator $\hat{\xi} : B(H) \to \mathcal{H}$ be such that $\hat{\xi}(p) = \xi(p)$, for all $p \in \mathcal{P}$. By Theorem 2.3 of Matvejchuk (1997), for all $h \in \mathcal{H}$ there exist a unique $M_h \in L_1$ and a unique number c_h, such that $\xi_h(q) := (\xi(q), h)_{\mathcal{H}} = \operatorname{tr}(qM_h) + c_h \dim(q_+ H)$, $\forall q \in \mathcal{P}$.

Let us assume for the moment that there exist $h_0 \in \mathcal{H}$ such that $c_{h_0} \neq 0$. For any $f \in \Gamma^+ \backslash S$ we have $f = \alpha e_+ + \beta e_-$, where

$$|\alpha|^2 - |\beta|^2 = 1, \qquad |\alpha| > 1, \tag{6}$$

and $e_\pm \in H^\pm \cap S$. In addition

$$p_f + (p_f)^* = p_f + p_{Jf} = [., f]f + [., Jf]Jf = 2(|\alpha|^2 p_{e^+} - |\beta|^2 p_{e^-}).$$

By the linearity of ξ,

$$\begin{aligned}\mathrm{tr}((p_f + (p_f)^*)M_{h_0}) + 2c_{h_0} &= (\xi(p_f), h_0)_{\mathcal{H}} + (\xi((p_f)^*), h_0)_{\mathcal{H}} \\ &= (\hat{\xi}(p_f + (p_f)^*), h_0)_{\mathcal{H}} \\ &= 2(|\alpha|^2\hat{\xi}(p_{e^+}) - |\beta|^2\hat{\xi}(p_{e^-})), h_0)_{\mathcal{H}} \\ &= 2(|\alpha|^2(\hat{\xi}(p_{e^+}), h_0)_{\mathcal{H}} - |\beta|^2(\hat{\xi}(p_{e^-}), h_0)_{\mathcal{H}}) \\ &= 2(|\alpha|^2(\xi(p_{e^+}), h_0)_{\mathcal{H}} - |\beta|^2(\xi(p_{e^-}), h_0)_{\mathcal{H}}) \\ &= \mathrm{tr}((p_f + (p_f)^*)M_{h_0}) + 2|\alpha|^2 c_{h_0}.\end{aligned}$$

Thus $2c_{h_0} = 2|\alpha|^2 c_{h_0}$. This contradiction with (6) proves that $c_h = 0$, for all $h \in \mathcal{H}$. Hence ξ is a w-linear measure. □

Theorem 1. *Let H be a Krein space,* $\dim H \geq 3$*, and let $\xi : \mathcal{P} \to \mathcal{H}$ be a bounded $\mathcal{H}$-measure.*

1) *If* $\dim H^+ < +\infty$ *then there exists a unique linear $\mathcal{H}$-measure ξ_l and a unique semiconstant ξ_s such that $\xi_l + \xi_s = \xi$;*
2) *If* $\dim H^+ = +\infty$ *then the $\mathcal{H}$-measure ξ is linear.*

Proof:

1) Let $\dim H^+ < +\infty$, $h \in \mathcal{H}$. By (1), $(\xi(q), h)_{\mathcal{H}} = \mathrm{tr}(qM_h) + c_h \dim(q_+H)$, for all $q \in \mathcal{P}$. By Proposition 2, the antilinear functional $h \to \mathrm{tr}(p_f M_h)$ $(= [f, f](JM_h f, f)_H)$ is continuous on $\mathcal{H}$, for all $f \in \Gamma$. The antilinear functional $h \to (\xi(q), h)_{\mathcal{H}}$ is continuous on $\mathcal{H}$, also. Hence the antilinear functional $h \to c_h$ is continuous on $\mathcal{H}$. By Riesz's theorem, there exists a unique vector $x_0 \in \mathcal{H}$ such that $c_h = (x_0, h)_{\mathcal{H}}$, for all $h \in \mathcal{H}$. The function $\xi_s(q) := \dim(q_+H)x_0$, for all $q \in \mathcal{P}$, is a semiconstant measure. It is clear that the function $\xi_l(q) := \xi(q) - \xi_s(q)$, $q \in \mathcal{P}$ is a linear bounded $\mathcal{H}$-measure.
2) Let $\dim H^+ = +\infty$. By Theorems 2.1 and 2.3 of Matvejchuk (1997), the $\mathcal{H}$-measure is w-linear measure. By Proposition 3, ξ is a bounded linear measure. □

Corollary 1. *Let H be a complex Krein space. Every $\mathcal{H}$-measure is bounded and linear if and only if* $\dim H^+ = \dim H^- = \infty$.

Proof: Let $\dim H^+ = \dim H^- = \infty$ and let ξ be an $\mathcal{H}$-measure. By Theorem 2.3 of Matvejchuk (1997) the complex measure ξ_h is linear. Hence ξ is w-linear. By Proposition 3, ξ is a linear measure.

Conversely, let every $\mathcal{H}$-measure on $\mathcal{P}$ be bounded and linear. Assume for the moment $\dim H^+ < +\infty$. Let ξ be a semiconstant measure. Every semiconstant measure is bounded and is not linear. We have a contradiction. □

Corollary 2. *Let* $3 \le \dim H$ *and let* ξ *be a bounded* $\mathcal{H}$*-measure such that* $\xi(p) = 0$ *for all* $p \in \mathcal{P}^-$.

1) If $\dim H^+ < +\infty$ *then* $\xi \equiv 0$ *or* ξ *is a semiconstant measure.*
2) If $\dim H^+ = +\infty$ *then* $\xi \equiv 0$.

Proof: By (1) and by $\xi(p) = 0$ for all $p \in \mathcal{P}^-$, we have $0 = (\theta, h)_{\mathcal{H}} = (\xi(p_f), h)_{\mathcal{H}} = \mathrm{tr}(p_f M_h) = -(JM_h f, f)_H$, for all $f \in \Gamma^-$. Thus $(JM_h f, f)_H = 0$, $\forall f \in S \cap \beta^{--}$. By Lemma 1, $\|JM_h\| = 0$, $\forall h$. Thus $M_h \equiv 0$. Hence $\xi_l \equiv 0$. By Theorem 1, $\xi = \xi_s$. □

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